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Students' Learning Trajectories in Understanding Linear Equations System with Three Variables for Vocational Students

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ABSTRAK

Penelitian ini bertujuan untuk mendeskripsikan kesesuaian *hypothetical learning trajectories* (HLT) dalam pembelajaran SPLTV yang dirancang dengan proses pembelajaran yang terjadi di kelas dan proses penalaran yang dilalui siswa dalam memahami SPLTV. Penelitian ini merupakan siklus pertama dari metode *design research* dengan tiga tahapan, yaitu *preliminary design*, *teaching expeirment*, dan *retrospective analysis*. Pengumpulan data dilakukan dengan wawancara guru dan siswa, observasi proses pembelajaran, studi dokumen jawaban siswa saat pembelajaran, dan analisis foto atau video yang diambil saat penelitian. Analisis data dilakukan dengan data triangulasi, yaitu membandingkan HLT yang dirancang sebagai patokan dengan pembelajaran aktual yang terjadi dengan melihat kembali hasil observasi, dokumentasi, jejak digital yang diambil, dan hasil wawancara yang dilakukan. Hasil penelitian menunjukan bahwa lintasan belajar siswa berkembang melalui tingkatan berpikir, mulai dari pemahaman konsep aljabar dasar hingga penemuan prosedur formal dalam menyelesaikan masalah SPLTV. Penelitian ini memberikan kontribusi pada teori HLT dengan menunjukkan efektivitasnya sebagai teori instruksional lokal yang memfasilitasi penemuan kembali secara mandiri melalui konteks yang spesifik pada sekolah vokasi atau kejuruan. Studi ini membuktikan bahwa HLT berfungsi sebagai jalur konjektur yang fleksibel untuk mengakomodasi tahap konstruksi pemahaman siswa sambil tetap mengacu pada hierarki kognitif yang diperlukan dalam menguasai konsep kompleks seperti SPLTV. Selain itu, penelitian ini menekankan bahwa kekuatan HLT terletak pada kemampuannya mentransformasi interaksi kelas menjadi interaksi horizontal, di mana model yang dihasilkan siswa menjadi sarana utama dalam membangun pemahaman konsep.

Kata Kunci: lintasan belajar; HLT; RME; penelitian desain; SPLTV.

ABSTRACT

This study aims to describe the alignment of hypothetical learning trajectories (HLT) in SPLTV designed with the actual learning processes that occur in the classroom, and the reasoning processes students go through in understanding SPLTV. This present study is the result of the first cycle of the design research method with three stages, namely preliminary design, teaching experiment, and retrospective analysis. Data collection was carried out through interviews with teachers and students, observation in the classroom, and study of students answer sheets during the teaching experiment, and analysis photos or videos taken during the research. Data analysis was conducted by using data triangulation, which involves comparing the HLT designed as a reference with the actual learning that occurred and reviewing the results of observations, documentation, digital footprints, and interviews. The research findings indicate that students' learning trajectories develop through levels of thinking, starting from understanding basic algebraic concepts to discovering formal procedures for solving SPLTV. This study contributes to HLT theory by demonstrating its efficacy as a local instructional theory that facilitates independent rediscovery of formal procedures from informal, vocational specific contexts. It proves that HLT functions as a flexible conjectural route that accomodates personal construction phases while maintaining the necessary cognitive hierarchy for mastering complex xoncepts like SPLTV. Furthermore, it highlights that the HLT's strength lies in transforming traditional classroom dynamics into horizontal interactions, where students' generated models become the primary vehicle for conceptual growth.

Keywords: learning trajectory; HLT; RME; design research; SPLTV.

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1. INTRODUCTION

Planning before teaching is a crucial point in the effective and high-quality implementation of teaching in the classroom (Cevikbas et al., 2024; Sari et al., 2025). Teaching with tiered or leveled learning objectives, in accordance with mathematical concepts, has proven to be more effective than teaching with only specific learning goals in mind (Baroody et al., 2024). Although the term learning trajectory sounds focused on solely learning rather than teaching, the definition of learning trajectory (LT) itself clearly emphasizes teaching and learning activities (Clements, 2011; Clements & Sarama, 2004; Mutaqin et al., 2023). The use of learning trajectory in mathematics education has attracted the attention of decision-makers, educators, curriculum developers, and educational researchers for several decades. This is because learning trajectory is a useful tool for providing guidance on standards, curriculum, lesson design, teaching, and educational research (Clements et al., 2020; Afriansyah & Arwadi, 2021; Nursyahidah, Albab, & Rubowo, 2025). The strength and uniqueness of a learning tracjectory are formed from the interconnection between two aspects, namely the progressive development of thinking abilities and a sequence of learning activities, both of which are focused on learning goals. However, a significant research gap exists in the context of vocational education, where instructional designs often prioritize the mechanical sequence of algebraic steps over the qualitative evolution of student reasoning. This creates a problematization where the activities are disconnected from the thinking abilities, leading to a failure in students' conceptual mastery of SPLTV. Without a validated HLT that explicitly bridges these two aspects, students frequently struggle to transition from informal situational logic to formal mathematical modeling. Therefore, this study addresses this void by designing an HLT that serves as a local instructional theory to synchronize activities with the actual cognitive trajectories of vocational students.

The modern learning trajectory concept developed by educators and psychologists is referred to as learning sequences or hierarchies (Baroody et al., 2009; Wood et al., 2006; Utari et al., 2025). Clements employs the term Hierarchic Interactionalism, which evolved into a more complex view of cognition and learning (Clements, 2011). He emphasizes that a learning trajectory includes children's levels of thinking, not just their ability to answer math questions correctly, but also how and why students think about a topic, including the cognitive actions on objects that influence the way they think. This situation will later demonstrate to students that a math problem can be solved differently by students with different levels or abilities of thinking, even though both can come up with the right answer. The primary goal of learning trajectory is the students' qualitative thinking processes, the mental and conceptual stages children go through, not jut the order of knowledge and skills that must be memorized or acquired (Fauzan & Diana, 2020; Tanggaard & Elmholdt, 2007; Prihandhika & Azizah, 2025).

Learning trajectory is student-centered, focusing on students' unique construction phases, which model how students might think differently, imperfectly, and perhaps incorrectly from an expert's perspective, but these are important stages in the development of their understanding (Baroody et al., 2024). Learning trajectory requires continuous, comprehensive, and simultaneous analysis of four interconnected components, namely goals (what students should learn), pedagogical tasks (activities or materials used in learning), teaching (how teachers deliver the material), and student thinking and learning (how students respond, comprehend, and build concepts) (Clements, 2011; Afriansyah et al., 2023). In their research, Wood, Williams, and McNeal (Wood et al., 2006) and Ellis et al. (2022) stated that these levels or stages indicate students' mathematical thinking, which is a mental activity involving abstraction and generalization of mathematical ideas.

Initiatives to define teaching processes or learning flows are not novel. However, more complex or spesific attempts to clarify students' actual cognitive development culminated to the development of hypothetical learning trajectory (HLT), which is a detailed form of LT (Clements & Sarama, 2004; Baroody, Clements, & Sarama, 2022) designed to be tested in mathematics classroom (Fauziyah, 2023). When studied closely, the hypothetical is commonly referred to as a conjecture, emphasizing that the trajectory is the best prediction of how students will develop in their understanding (Simon & Tzur, 2004; Pêgo, Miguéis, & Soeiro,2024). Meanwhile, learning trajectory emphasizes the qualitative thinking stages of students, not just the logical sequence of skills. HLT consists of (Baroody et al., 2009; Simon, 1995; Simon & Tzur, 2004; Callejo et al., 2022) a combination of three components, namely (a) meaningful learning goals; (b) task, activities, or problems aimed at achieving the learning goals; and (c) assumptions about students' learning. Although HLT is a comprehensive tool form classroom design, its application to SPLTV is particularly critical. This is because SPLTV represents a high-stakes transition in secondary mathematics where students often lose conceptual footing due to the abrupt shift from basic linear equations to multi variable siystems. A specific HLT is therefore required to identify the unique cognitive hurdles of this topic and to provide a scaffolded pathway that ensures students do not merely replicate procedures but truly understand the systemic nature of the equations.

The presence of HLT has also been proven to support the develpoment of students' conceptual understanding, confidence in reasoning abilities, and communication (Umasugi et al., 2022). Nevertheless, in practice, most of the learning is not structured based on students' thought trajectories. Learning is based on existing learning achievements, with an emphasis on students' mathematical knowledge and understanding (Sarumaha & Rizkianto, 2025). In reality, the problems used in class are more stody-based than problems related to life in general (Sari et al., 2022). Not only are problems at the heart of learning, but the way teachers teach additionally encourages students to think school mathematics as something distinct from life (Sarumaha, 2016). These impacts solving problems, which always focuses on the amount of information present in the question and the correct answers generated, rather than their significance in life. Furthermore, interaction in the classroom is also limited to the teacher asking questions and students providing answers. This traditional form or pattern of interaction is also known as the IRE (initiate, response, and evaluate) pattern (Galen & Eerde, 2019). The teacher initiated the interaction with a question, the students provided answers, and the teacher evaluated by saying whether the answers were correct or not.

Given these learning constraints, it is necessary to design a learning path or trajectory for students that can guide or provide insight into the phases of student thinking that take place when understanding mathematical content. Besides designing the LT to be used in teaching, the form of mathematics learning used also needs to be considered. Learning mathematics should provide an environment that supports students to actively participate in investigating problems given by the teacher. Thorugh classroom instructions, students can explore new topics or relationships between topics, with an emphasis on understanding and knowledge (Galen & Eerde, 2019). Meanwhile, the problems proposed are mearningful and interesting that can stimulate students to think. Good problems (Galen & Eerde, 2019; Hoogland et al., 2018; Prudencio, Maximo, & Colombini, 2023) are open problems that cannot besolved using standard procedures alone, and therefore invite students to find their own answers, doing fundamental mathematics in solving the problems.

These characteristics can be found in Realistic Mathematics Education (RME) or Indonesian Realistic Mathematics Education (PMRI), which promotes mathematics learning with guided reinvention (Freudenthal & Reidel, 1983; Afriansyah, 2022). In the vocational education framework, this principle takes on a distingushing role by grounding mathematical abstraction in professional practice. Rather than using generic word problems, this study utilizes vocation-specific scenarios, such as culinary measurements or productuon cost analysis, as the primary design heuristic for developing learning activities within the HLT. The theory of Realistic Mathematics Education (Freudenthal & Reidel, 1983; Van Den Heuvel-Panhuizen & Drijvers, 2014; Fauzan, Nasuha, & Zafirah, 2024) provides opportunities for design heuristics in developing learning activities within an HLT. This approach bridges the gap between theoretical algebra and workplace application, ensuring that the mearningful context is directly relevant to the students' future careers. In this setting, the teacher's role shifts from a traditional lecturer to a facilitator who stimulates students to rediscover SPLTV concepts using their own intuition. By helping students formulate ideas based on vocationally situated problems, the teacher enables them to understand the thinking of their peers while simultaneously validating the utility of mathematics in their specific field of expertise.

In this study, we utilized knowledge from the literature on Linear Equations Systems in Three Variables (SPLTV) and applied this knowledge to learning progressions to design and evaluate learning trajectories in 10th grade students at Vocational High School Nurul Iman Bantul. Unlike traditional approaches that focus on meorization, this research designs a learning path specifically for the 10^{th} grade Vocational School (SMK) context, using problems relevant to students' major such as culinary and production recipes. Besides, the study provides empirical evidence of how students' qualitative thinking develops through specific levels – from basic algebraic concepts to the discovery of formal elimination and subtitutions procedures. Learning is conducted using RME as the learning model. Some aspects that are desired to be known include (1) how and to what extend does the designed hypothetical learning trajectories in SPLTV align with students' actual cognitive trajectories, and what specific factors account for any observed deviations between the hypothesis and the classroom reality? (2) how is the reasoning process students go through in understanding SPLTV?

2. METHOD

In line with the research questions posed, the method employed in this study is design research. In design research, research is conducted to develop theories about how the learning process of a specific mathematical concept and its instructional design support that learning (Gravemeijer et al., 2013; Gravemeijer & Cobb, 2006). Design research consists of three phases, namely preliminary design (preparation before the research is conducted), teaching experiment (implementation of the prepared learning design), and retrospective analysis (analysis of the results of the research conducted). The initial phase includes two activities, a literature review and interviews with teachers and students which serve as the basis for designing the HLT. In the next phase, the HLT that has been designed is implemented in the research. Students' work and reasonings in learning were observed and collected. In the final phase of design research, all data is analyzed by comparing the designed HLT with the students' actual learning processes. All possible causes were analyzed, and all possibilities were synthesized to refine the HLT for the next cycle. The research findings discussed here are from the initial of the first cycle of the overall series of studies conducted.

This research was conducted in the classroom of Vocational High School Nurul Iman Bantul during the first semester of the 2024/2025 academic year. The research subjects consisted of 15 tenth-grade students. Qualitative data in the study were obtained from observation, interviews, documentation, and digital evidences (Creswell & Creswell, 2018), all of which were prepared by the researchers before the study began. The primary method of validation is data triangulation. This involves comparing several distinct types of data to see if they support the same conclusions. The designed HLT, as a benchmark, will be compared with the actual learning that occurred in the classroom, while synthesizing information will be done by from teacher and students' interviews, classroom observations, and document analysis of students' answer sheets. Besides, digital evidence, such as photos and videos taken during the research cycle will be reviewed to verify the reasoning processes observed. Validation was also carried out by expert validation, which involved seeking advice or input on the instruments that had been created before those were tested in the field.

Data analysis in the study was conducted by comaring the HLT designed as a benchmark with the actual learning that occurred (Sarumaha et al., 2018) based on students' reasoning and understanding processes in learning SPLTV. Interviews with teachers and students were conducted at the beginning and end of the study. At the beginning of the study, the interviews were intended to gather information about students' mathematical reasoning and understanding abilities, the difficulties experienced by both teachers and students in learning. At the end of the learning, interviews with students were conducted to confirm various things that could not be concluded solely by looking at students' written test results or observations made during the learning process. In other words, not only did the researchers delve into the reasoning processes that took place in stduents while learning, but they also verified the results they had obtained, both through document analysis of students' work and photos or videos taken during the research.

3. RESULT AND DISCUSSION

The design of the HLT provides a conjectural route using a series of educational activities that assist students in attaining the learning objectives that have been set. Albeit, learning is a personal process, particular to each student, HLT aims to outline every possible learning path taken by all students (Gravemeijer, 2004), which requires empirical validation.

a. Research Findings

AS previously mentioned, in this present study, the researchers use RME as a theory that provides design heuristic for developing learning activities in LT. First, learning activities must be determined within a context that supports students to quickly engage and develop interconnected mathematical concepts. In this case, learning activities support progress in understanding important concepts in line with the learning objectives in LT. Second, activities must be structured to encourage students in developing a model of concrete mathematical objects and their relationships (Dijke-Droogers et al., 2022).

The following is the Hypothetical Learning Trajectory (HLT) used in this study,

Table 1. Designed HLT

Aktivities	Learning Goals	Activities Description	Hypothesis of Students' Solutions
Problem Representations	(1) Students can identify three unknown variables from the given problems; (2) Students can organize the data they obtain into an easy-to-understand format as they seem fit.	Given a production recipe with specific ingredients, students are asked to record the raw material requirements for each cake, including flour, sugar, and eggs, as well as the toatl production cost. Their task is to determine the unit cost per kilogram of each raw material.	(1) Students have difficulty separating variables and recording all the information provided in an unstructures narrative format; (2) Students successfully created simple tables to organize data vertically or horizontally, but they have not used algebraic symbols yet; (3) Students immediately tried to write mathematical sentences per raw material, for example flour 1 + flour 2 +flour 3 = total flour .
Exploring Informal Strategies	Students can use logical reasoning to simplify the number of variables without using formal algebra.	Students are asked to take a closer look at Recipe A and Recipe B. If Recipe A and Recipe B use the same ampunt of sugar and eggs, but different amounts of flour, then the difference in the cost of making the cake between Recipe A and B will only be affected by the difference in determining the unit price of flour.	(1) Students are just trying things out (trial and error) by randomly assigning prices to raw materials without considering other variables; (2) Students use subtraction and elimination to find the price of one material, then use that price to subtitute for the variable value in another equation; (3) Students can explain that the strategy they used was to find difference between the two total costs to eliminate the influence of the prices of the other two raw materials.

Aktivities	Learning Goals	Activities Description	Hypothesis of Students' Solutions
Transformation to a Formal Model	(1) Students can formally write down a system of linear equations with three variables; (2) Students can restate the elimination and subtitution steps to solve problems.	The teacher facilitates the discussion. Students are asekd to simplify the raw materials in the recipe using corresponding variables. Students then wrote down three linear equations from the tables or representations they had created in Activity 1. Students revisited the startegies they used in Stage 2, and with teacher guidance, reduced the production cost of Recipe A by Recipe B, which is the same as subtarcting equation 1 from equation 2 to eliminate a specific variable.	(1) Students have difficulty consistently subtituting variables and frequently make mistakes in placing coefficients; (2) Students can correctly write down the SPLTV and succesfully perform one formal elimination step to obtain a system of linear equations with two variables; (3) Students can explicitly compare the informal process with formal procedures and are able to explain why elimination results in equivalent equations (by elimintaing one raw material).
Implementation of Formal Procedures	Students can systematically solve system of linear equations with three variables (SPLTV) using elimination and subtitution methods and interpret the results in a context	Students apply formal steps (elimination to a system of two linear equations, solving the system of two linear equations, and back-subtitution) to solve the new system of three linear equations. After obtaining the values of $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$, they were required to convey to the teacher and the entire class	(1) Students often make mistakes in multiplying or adding coeficients during elimination or subtitution, and forget to reconsider the final results; (2) Students successfully completed all formal procedures and achieved the correct solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$; (3) Students not only discover the solution, but are also able to validate the results by substituting the values of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ into the three original equations. They can also

Aktivities	Learning Goals	Activities Description	Hypothesis of Students' Solutions
	relevant to their major.	what the significance of the results they obtained was.	comment on the price's feasibility in the context of business as well as culinary arts.

The initial phase of the lesson served as the establishment of a meaningful situational context, where the teacher's detailed introduction acted as a scaffold to ensure accessibility, a core principle of RME. By inviting clarifying questions before group work, the teacher initiated a socio-mathematical norm where understanding the problem's constraints is prioritized over immediate calculation. The transition to individual reflection before group discussion is a strategic pedagogical move to facilitate internalization. This pre discussion activity ensures that each student engages in horizontal mathematization, allowing them to form independent mental models before being influenced by peer logic. Within the small groups, the interaction shifted toward collective reinvention, the sharing of solutions was not merely a peer-tutoring session but a negotiation of efficient startegies. This process exemplifies the interactivity principle, where the diverse speeds of student recognition provide the cognitive conflict necessary for level-raising. The more advanced students' quick realizations served as catalysts for classroom discussions, forcing the articulation of informal arguments into more structured mathematical reasoning. In this context, the varied pace of understanding was not a pedagogical hurdle but a resource for social mediated learning, allowing the teacher to guide students toward a shared formalization of SPLTV concepts.

Figure 1. Students working in Groups

The teacher went around monitoring the students' discussions, soliciting some groups to lay out the methods they would use to solve the problem. The goal was to ensure that all students grasped what was being requested and to help the groups get started on their work. Moving around the class from one group to another also helped the teacher examine the results of the group work and select some groups with different answers. At the end of the discussion, the teacher invited some groups to come to the front of the class to talk about their findings. The other groups raised issues to the presenting groups and gave what they thought.

Figure 2. Students' Presentation

Unlike the vertical interaction that typically takes place in the classroom, where interaction is limited only between the teacher and one student, in this study, the interaction that occurs is not only between the teacher and the students, but also among the students, which is referred to as horizontal interaction. This kind of interaction also found by Galen and Eerde in their study in which conducting RME in the classroom supports interactions in any ways, as also one of primary characteristics of RME (Heuvel-Panhuizen, 2020), from teacher to students and among students (Galen & Eerde, 2019). The theoretical contribution of this interaction pattern lies in its role as a catalyst for level-raising. When students engage horizontally, they are forced to externalize their informal model of a situation, making these models subject to peer critique and refinement. This social negotiation serves as the bridge that transforms individual, situational intuition into a generalized, formal model for mathematical reasoning. Thus, horizontal interaction is not merely a classroom management choice, but a critical epistemic mechanism within HLT theory that allows for the collective construction of formal SPLTV procedures, proving that mathematical authority can be successfully redistributed to the student collective without losing conceptual rigor. The teacher provides students with the opportunity to formulate their ideas while attending to what they say, especially in conducting large class discussions, which are the core of mathematics learning. Teachers play a significant part in reframing students' proposals and ensuring that discussions highlight essential concepts, which are of course derived from the students themselves as the startting point for learning.

b. Discussion

Based on the designed HLT, in the first activity, where students were introduced to contextual problems and represented the rpoblems based on the information in the problem, most students still experienced difficulties. Students have been able to identify variables that can be used to represent the problem, but they have failed to organize the data in a structured manner. As the teacher went around monitoring discussion, the students still needed the teacher' s help in representing the problem in the form of a table. Only a few students in the group were able to create structured informal representations. The students created a neat table containing coeficients and constants, but still used abbreviations based on the problem information. In the second activity, where students were supported to explore infromal strategies, students were given two cookies' recipes and determined the raw materials that caused differences in the total cost of the baking model. Some students intuitively grasp the principle of elimination. This demonstrated the success of RME in facilitating the reinvention of mathematical concepts through situations (Heuvel-Panhuizen, 2020). However, there are still groups of students who already understand the concept of proportion but are not yet able to apply the elimination principle, or in solving problems, students do not pay attention to the coefficients following the variables.

The third activity focused on the transformation from informal representations to a formal mathematical model, specifically requiring students to translate contextual information into a system of three linear equations using variables. While the students demonstrated proficiency in the initial symbolization, indicating that translating the problem into a formal algebraic form was not a significant barrier, the actual execution of the elimination process revealed a cognitive gap. Many students experience donfusion when anvogating formal elimination or complex algebraic manipulations. This suggests that while the model of (the equation) was easily constructed, the transition to using it as a model for mathematical reasoning required more intensive support. However, the success of several groups in effectively formalizing the elimination procedure during their presentations confirms that the HLT successfully facilitated the intended learning path. Notably, one group' s ability to reduce the system into linear equation system of two variables served as a critical teaching moment. Rather than providing the solution directly, the teacher used this student-generated outcome as a pivot point to stimulate slass-wide reflection on the logic of the elimination method. This interaction highlights a shift in the teacher' s role from a source of knowledge to a facilitator of horizontal interaction, where theie findings become the promary vehicle for collective conceptual growth.

The extended duration required for the fourth activity signifies a critical transition from intertwined model to formal mathematical procedures. In the RME framework, the students' struggle to define their steps represents the process of instrumenttaion, where algebraic symbols move being objects to be manipulated to becoming tools for systemic in solving problems. The time investment allowed for development of relational understanding rather than mere instrumental proficiency, students were not just seeking an answer but were negotiating the meaning behind the reduction of variables. Figure 3 is an example of one of the answers produced by students in their group. It appears that the students have arrived at the correct solution and validated the results which indicates a successful transition to formal mathematization. By validating their own results, students moved beyond the model for stage into the realm of formal algebraic reasoning, where the system of equations is understood as a unified mathematical entity rather than a series of disconnected steps. Students demonstrate the ability to perform algebraic manipulation, verification, and interpretation, connecting the formal mathematical results obtained to the context being discussed. Even though some groups still encountered diffculties in solving the problem due to incorrect algebraic manipulation, class discussions helped the groups arrive at the correct meaning and answers to the problem. This confirms that the slowing down in the implementation phases is a necessary cognitive requirement for ensuring that formal procedures are grounded in conceptual clarity.

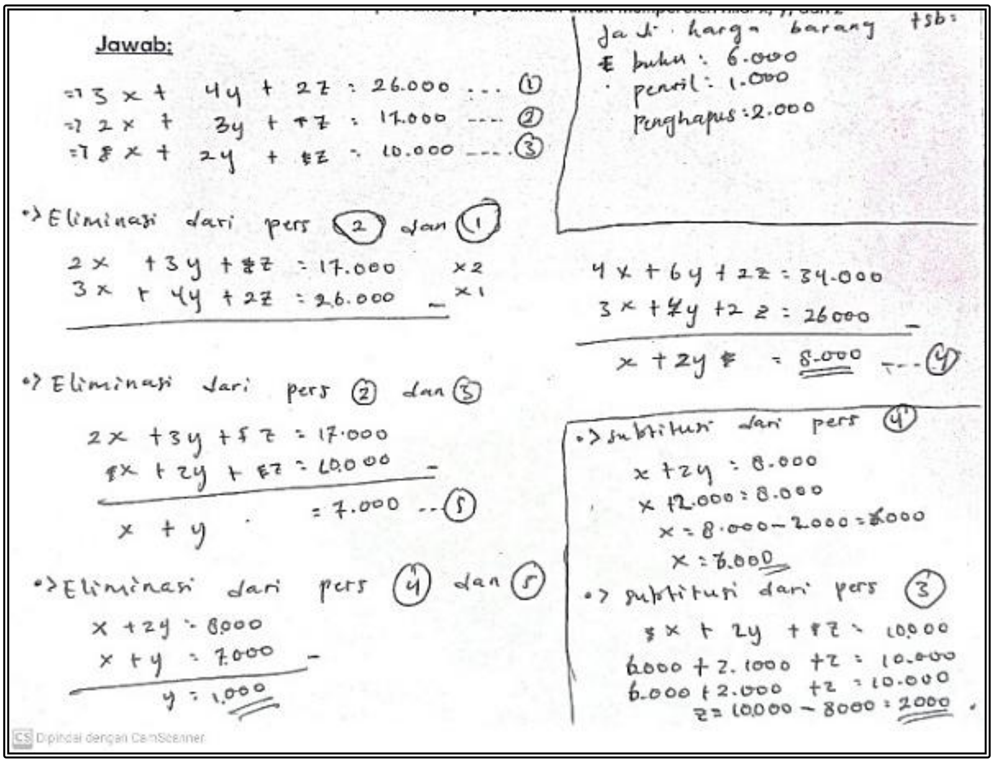

Figure 3. One of Groups' Answer to Activity 4

Based on the outcomes of all activities carried out, the HLT created is in alignment with students' reasoning in learning SPLTV. The chosen problem situation or context is relevant to the students' major, so all students can relate to what is presented in the question. The learning objectives of each activity successfully guided students from initial data to informal models (mathematical representations or models that emerged from the students), demonstrating a process from model of to model for in solving problems, in accordance with the stages of model development in RME (Van den Heuvel-Panhuizen, 2003). This is one of the most notable characteristics of LT compared to other learning plans because of the unique student construction that is designed to memerge from the students themeselves. Hypotheses about students' thinking processes are not merely preparatory tools, they constitute a fundamental theoretical component of HLT that bridges the gap between instructional design and cognitive development. By predefining potential learning path, the HLT serves as a conjectural framework that enables teachers to anticipate various levels of student abstraction. The theoretical contribution here lies in how the HLT transforms pedagogical intervention from a reactive measure into a proactive scaffolding strategy. By analyzing student responses across different levels, ranging from informal sitautional models to formal algebraic structures, the teacher utilizes the HLT to determine the precise timing for level raising. This ensures that interventions do not bypass the students' own construction of meaning but rather support the guided reinventuon of SPLTV concepts. Consequently, this study demonstrates that HLT contributes to theory by acting as a dynamic navigational map for teachers, allowing them to maintain the balance between student discovery and the systematic attainment of formal mathematical goals.

According to the data collected, HLT, which includes a combination of learning objectives, activities or problems used, and hypotheses about students' learning processes, is a flexible design employed by teachers to anticipate students' thinking, which sometimes provides a distinctive viewpoint (Ivars et al., 2018). Students do not immediately comprehend the procedure for solving SPLTV using specific formulas, but rather over several stages. It begins with grasping basic algebraic concepts such as using varibels and corresponding coefficients before reaching to solve problems. Students are able to construct their own reasoning, which may differ from existing conventional approaches, but is logically sound within their context and abilities. Students' reasoning grows from recognizing real-world situations to symbolic abstraction, with each stage of reasoning they go through clarifying why and how they took the step. This aligns with the research conducted by Zahner and Wynn, which found that research employing learning trajectories investigates how mathematical activities work, along with teacher-student interactions in the classroom, which influence students' reasoning (Zahner & Wynn, 2023).

The role of the teacher in this study shifted from telling and explaining to asking questions, listening, and trting to understand how students think or reason, as well as guiding their learning process. Meanwhile, the role of the students, as mentioned by Galen and Eerde (Galen & Eerde, 2019) in their research, changed from passively listening and trying to give the correct answers to being more active in asking questions and participating in discussions on how to solve problems. Students think and speak as much as their teachers. In simple terms, in this class, the focus of learning is on the students. The teacher only acts to present the problem, then gives the students time, and stimulates them to think. The teacher will centainly provide assistance or support by answering technical or informative questions and offering some guidance or help, but they will not provide a solution.

4. CONCLUSION

Based on the data analysis and discussion presented, the following conclusions might be reached regarding the use of Learning Trajectories (LT) and students' reasoning processes. The HLT designed is an effective tool for understanding mathematical concepts since it focuses on students' thinking stages. Albeit the learning paths students take were not exactly the same as the designed HLT, the HLT is still able to accommodate the construction of students' understanding that emerges during the problem-solving process. LT not only supports the principle of students' reinvention but also facilitates the processes that take place. Students' ways of thinking follow a learning hierarchy where mastering initial skills is a prerequisite for solving subsequent problems. Students are able to move from informal strategies to formal algebraic models throughout a series of thinking phases.

While this study provides valuable insights into the learning trajectories of SPLTV, several limitations must be acknowledged. First, as this research represents the first cycle of a design research process, the findings are formative and primarily focus on the initial alignment of the HLT. The small size of 15 students in a specific vocational context means that while the results offer deep qualitative insights, they may not be immediately generalized to all vocational settings without further iterative cycles. Additionally, the study focused heavily on the transition from informal to formal models, learving the long-term retention of these formal procedures and their apllication to non routine professional problems as a subject for future longitudinal investigation.

Beyond its practical application, this study offers significant theoretical implications for the development of local instructional theories. It explicitly demonstrates that HLT serves as more than a predictive tool, it acts as an epistemic bridge that synchronizes the logic of the subject (SPLTV) and the logic of the learner. The research contributes to HLT theory by proving that horizontal interaction is a critical mechanism for learning better, where social negotiation among peers facilitates the transition from models of to models for. Furthermore, by situating RME within vocational framework, this study expands the scope of guided reinvention theory, showing that the professional context does not merely provide cover stories for math problems but serve as foundational cognitive anchors that allow students to rediscover complex algebraic sustem independently. This suggests that HLTs in mathematics education must be inherently domain specific and context sensitive to effectively support the qualitative construction of formal mathematical knowledge.

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